MATH1031

DISCRETE MATHEMATICS

Epiphany 2025–2026

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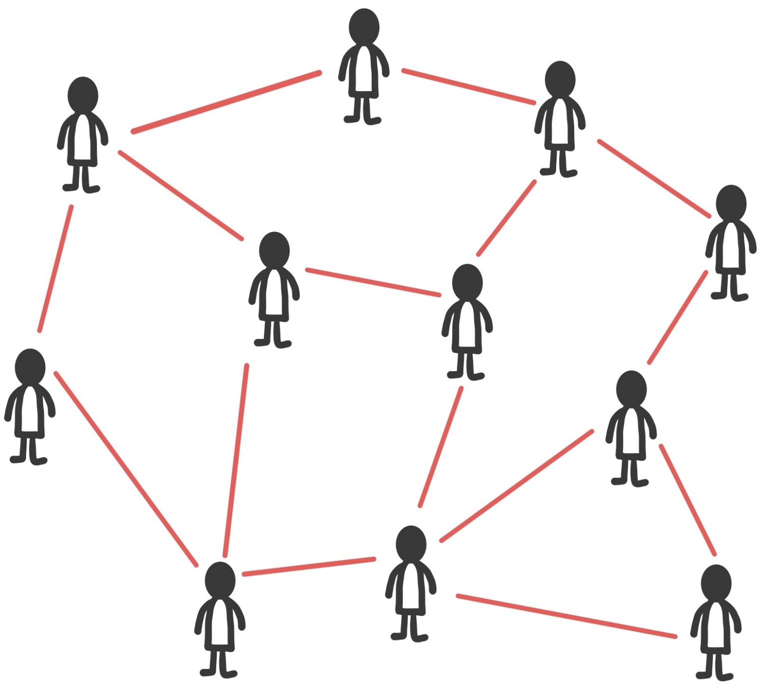

"Graph theory is the study of relationships."

— Paul Erdős

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Chapter 8. What is a graph?

8.1 Basic definitions

Informally, a **graph** consists of a set of **vertices**, represented graphically by points, together with **edges**, represented by lines, that join certain pairs of vertices, as in the following figure:

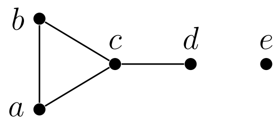

Figure 8.1: Example of a graph with vertex set $V = \{a, b, c, d, e\}$, and edge set $E = \{ab, ac, bc, cd\}$.

Definition 8.1.1. *A (general) graph $G = (V, E)$ consists on a pair of sets (V, E) where V is the **vertex set** and $E \subset V^2 = V \times V$ is the **edge set**.*

Remark 8.1.2. Since $V \times V$ is a set of ordered pairs, the edges of the graph defined above are ordered pairs of vertices. More precisely, for $u, v \in V$:

- The pairs (u, v) and (v, u) are different. In this case, the edge (u, v) may be interpreted as an edge from u to v; such edges are called *directed edges*. A graph containing directed edges is called a *directed graph*.
- The pairs (u, u) also belong to $V \times V$. These would correspond to edges that start and end at the same vertex, and are called *loops*.

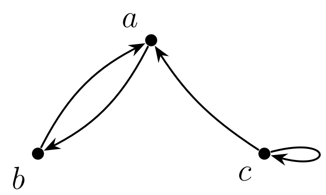

Figure 8.2: Example of a directed graph with a loop. The vertex set is $V = \{a, b, c\}$ and the edge set is $E = \{(a, b), (b, a), (c, a), (c, c)\}$. Notice that (a, b) and (b, a) are different edges and are represented by arrows in opposite directions.

! In this course we will **not** consider directed graphs nor graphs with loops.

In order to exclude directed edges and loops, we should take our edge set from a different collection. For V be a vertex set, let

$$\binom{V}{2} := \Big\{ \{u, v\} : u, v \in V,\ u \neq v \Big\}, \tag{8.1}$$

be the set of *unordered* pairs of *distinct* vertices.

Remark 8.1.3. Recall that (u, v) denotes an *ordered* pair of vertices, so that in general $(u, v) \neq (v, u)$. By contrast, $\{u, v\}$ denotes an *unordered* pair of vertices, and hence

$$\{u, v\} = \{v, u\}.$$

This distinction is the reason why taking $E \subset V \times V$ leads naturally to directed edges, whereas taking $E \subset \binom{V}{2}$ produces undirected edges.

Definition 8.1.4. *A* ***simple graph*** $G = (V, E)$ *consists on a pair of sets* (V, E) *where* V *is the vertex set and* $E \subset \binom{V}{2}$ *is the edge set. In other words, a simple graph is an undirected graph with neither loops nor multiple edges.*

$\longrightarrow$ From now on, we will work **only** with simple graphs.

Let us now set up some basic definitions in order to study the structure of a graph and its components:

Definition 8.1.5. *Let* $G = (V, E)$ *be a simple graph.*

- *If* $e = \{u, v\} \in E$*, then the vertices* u *and* v *are called the* ***endpoints*** *of the edge* e*. We also say that the edge* e *is* ***incident*** *with each of its endpoints.*
- *Two vertices* $u, v \in V$ *are said to be* ***adjacent*** *if* $\{u, v\} \in E$*. In this case, we can also say that* u *and* v *are* ***neighbours****.*
- *A vertex* $v \in V$ *is called* ***isolated*** *if it is not adjacent to any other vertex, that is, if there is no edge* $e \in E$ *incident with* v*.*

Remark 8.1.6. Since we will only work with simple graphs, edges are unordered pairs of vertices. Accordingly, we will often write $\{u, v\}$ simply as uv when referring to an edge.

Example 8.1.7. In Figure 8.1, vertices a and b are adjacent, and hence are the endpoints of the edge $\{a, b\}$ (denoted by ab, with $ab = ba$), while vertex e is isolated.

So far, we have described graphs by specifying their vertex and edge sets, or by drawing them. However, it is useful to encode the adjacency information of a graph using a matrix:

Definition 8.1.8. *Let* $G = (V, E)$ *be a simple graph with vertex set* $V = \{v_1, v_2, \ldots, v_n\}$*. The* ***adjacency matrix*** *of* G *is the* $n \times n$ *matrix* $A = (a_{ij})_{1 \le i,j \le n}$ *whose entries record the edge set* E*, and are defined by*

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

The (i, j)-entry of the adjacency matrix records whether the vertices v_i and v_j are adjacent. Since G is a simple graph, the adjacency matrix has zeros on the diagonal and is symmetric.

Example 8.1.9. Consider the graph shown in Figure 8.1, with vertex set $V = \{a, b, c, d, e\}$. The adjacency matrix of this graph is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the rows and columns are indexed by the vertices a, b, c, d, e in that order. Notice that the matrix is symmetric, reflecting the fact that the graph is undirected, and has zeros on the diagonal since the graph is simple. Moreover, the final row and column consist entirely of zeros, indicating that the vertex e is isolated.

Remark 8.1.10. Once an ordering of the vertex set is fixed, the adjacency matrix completely determines the graph G, and may therefore be regarded as a matrix representation of the graph.

We now introduce the notion of the **degree** of a vertex, which counts its neighbours:

Definition 8.1.11. *Let* $G = (V, E)$ *be a simple graph and let* $v \in V$*. The* ***degree*** *of the vertex* v*, denoted by* $d_G(v)$ *(or simply* $d(v)$*, or even* d_v*), is the number of edges in* E *that are incident with* v*. Equivalently, for a simple graph, the degree of a vertex is the number of vertices adjacent to it.*

Remark 8.1.12. If A is the adjacency matrix of a simple graph G, then the degree of a vertex v_i in G is equal to the sum of the entries in the ith row of A:

$$d_G(v_i) = \sum_{j=1}^{n} a_{ij}.$$

Definition 8.1.13. *A vertex of degree* 1 *in a graph is called a* ***leaf****.*

Example 8.1.14. In the graph shown in Figure 8.1, we have

$$d_G(a) = 2, \quad d_G(b) = 2, \quad d_G(c) = 3, \quad d_G(d) = 1, \quad d_G(e) = 0.$$

In particular, the vertex d is a leaf, while the vertex e is isolated.

8.2 Some standard simple graphs

In this section, we describe some standard families of simple graphs. Recall that for any set S, its cardinality is denoted by $|S|$ and tells us how many elements there are in the set S.

We begin with the **empty** or **null graph** on m vertices, denoted by N_m. This graph, with $|V| = m$ vertices but $|E| = 0$ (no edges), is not the most exciting one.

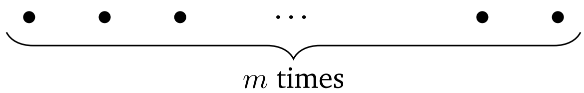

Figure 8.3: The Null graph N_m.

The **complete graph** on n vertices, denoted by K_n, consists of n vertices with an edge joining every pair of distinct vertices. In this case, $|V| = n$ and $|E| = \binom{n}{2}$.

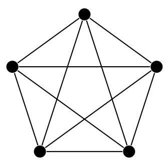

Figure 8.4: The complete graph K_5.

The **cycle graph** on n vertices, denoted by C_n, has vertex set $V = \{v_1, \ldots, v_n\}$ and edge set

$$E = \{v_1v_2,\ v_2v_3,\ \ldots,\ v_{n-1}v_n,\ v_nv_1\}.$$

In other words, the vertices can be arranged in a ring, with each v_i joined to v_{i+1}, and with v_n joined back to v_1, forming a *cycle*. Thus C_n has $|V| = n$ vertices and $|E| = n$ edges, and every vertex has degree 2.

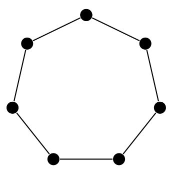

Figure 8.5: The Cycle graph C_7.

The **star graph** on n vertices, denoted by S_n, consists of one central "hub" vertex h (see Figure 8.6) joined to each of the remaining $n-1$ vertices. These $n-1$ vertices are leaves, each of degree 1, while the hub has degree $n-1$. Thus S_n has $|E| = n-1$ edges in total.

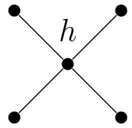

Figure 8.6: The star graph S_5.

The **wheel graph** on n vertices, denoted by W_n, can be constructed from the star graph S_n by adding edges joining the leaves of S_n in a cycle. Thus W_n has $|V| = n$ vertices and $|E| = 2(n-1)$ edges.

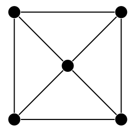

Figure 8.7: The wheel graph W_5.

The n-**cube** is denoted Q_n and has 2^n vertices. We can construct Q_n by labelling its vertices with n-digit binary strings, and by joining two vertices with an edge precisely when their binary representations differ in exactly one position. Thus Q_n has $|V| = 2^n$ vertices, and the number of edges is $|E| = n\,2^{n-1}$: there are n possible positions in which the strings can differ, and 2^{n-1} choices for the remaining positions.

For example, in Q_3 (see Figure 8.8 below), the vertices 001 and 011 are joined by an edge, since their binary representations differ in exactly one position (the second), whereas 001 and 111 are not joined, as they differ in two positions.

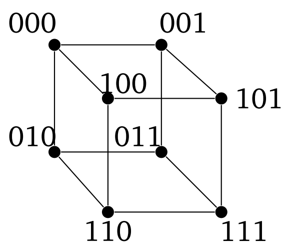

Figure 8.8: The Cube Q_3.

8.3 Handshaking

The main goal of this section is to answer the following question.

In how many different ways can we count the total number of handshakes at a party?

The answer to this question leads to a basic but extremely useful result in graph theory, known as the *Handshaking Lemma*:

Lemma 8.3.1. *For any graph $G = (V, E)$, we have*

$$\sum_{v \in V} d(v) = 2|E|\,. \tag{8.2}$$

Proof. Each edge has exactly two endpoints. When we sum the degrees of all vertices, each edge is counted once for each of its endpoints, and hence contributes 2 to the total. Since there are $|E|$ edges, we obtain

$$\sum_{v \in V} d(v) = 2|E|.$$

□

We have now answered the previous question: there are two ways to count the number of handshakes at a party. Counting handshakes one by one corresponds to the right-hand side of (8.2), while asking each person how many hands they shook corresponds to the left-hand side.

From the Handshaking Lemma we can deduce the following:

Corollary 8.3.2. *In any graph, the number of vertices of odd degree is even.*

Proof. Let $|V| = n$ and $|E| = m$, and let k be the number of vertices of odd degree. We wish to show that k is even. By the Handshaking Lemma ,

$$\sum_{v \in V} d(v) = 2m,$$

which is even. The sum of the degrees of the $n - k$ vertices of even degree is a sum of even numbers, and hence is even. Therefore, the sum of the degrees of the remaining k vertices of odd degree must also be even.

A sum of odd numbers is even only if there are an even number of terms. Hence k, the number of vertices of odd degree, is even. □

8.4 Graph isomorphism

Graphs are often represented by drawings, but the same graph can be drawn in many different ways: vertices may be placed in different positions on the page, and edges may be drawn using different curves. As a result, two drawings may look quite different while representing the same underlying graph. Conversely, two graphs may look similar at first glance, yet represent genuinely different structures.

For example, do the graphs in Figure 8.9 below represent the same graph?

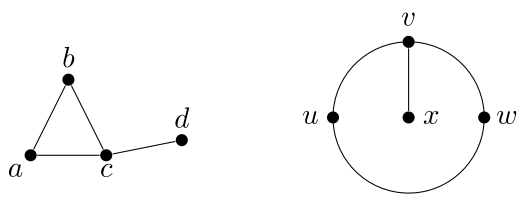

Figure 8.9: Two drawings of graphs. We denote by G_1 the graph on the left, with vertex set $V_1 = \{a, b, c, d\}$, and by G_2 the graph on the right, with vertex set $V_2 = \{u, v, w, x\}$.

In general, we ask ourselves the following question:

How can we decide whether two graphs are essentially the same, or genuinely different?

! The answer should not depend on how a graph is drawn, nor on the particular labels of its vertices.

To answer this question in a precise way, we need a notion that captures the structure of a graph independently of how it is drawn or labelled. This is the idea behind the concept of **graph isomorphism**:

Definition 8.4.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. We say that G_1 and G_2 are* ***isomorphic****, and write $G_1 \simeq G_2$, if there exists a bijection $\varphi : V_1 \to V_2$ between their vertex sets that* ***preserves adjacency****, that is, such that*

$$\{u, v\} \in E_1 \iff \{\varphi(u), \varphi(v)\} \in E_2.$$

In particular, if $G_1 \simeq G_2$, then the graphs must have the same number of vertices, $|V_1| = |V_2|$, and the same number of edges, $|E_1| = |E_2|$. More importantly, it must be possible to relabel the vertices of G_1 so that its edges correspond exactly to the edges of G_2.

Example 8.4.2. Let us show that the graphs in Figure 8.9 are isomorphic. We begin by noting that both graphs have the same number of vertices, $|V_1| = |V_2| = 4$, and the same number of edges, $|E_1| = |E_2| = 4$.

We now seek a bijection between the vertex sets that preserves adjacency. Degree considerations are often a useful first step when searching for an isomorphism. In both graphs, there are two vertices of degree 2, one vertex of degree 3, and one leaf (of degree 1). It is therefore natural to map first the vertices with unique degrees; we set $\varphi(d) = x$ and $\varphi(c) = v$. We may then $\varphi(a) = u$ and $\varphi(b) = w$.

Finally, let us check that this function ϕ does indeed preserve adjacency:

$$\begin{aligned} ab &\mapsto \varphi(a)\,\varphi(b) = uw\,, \\ ac &\mapsto \varphi(a)\,\varphi(c) = uv\,, \\ bc &\mapsto \varphi(b)\,\varphi(c) = wv\,, \\ cd &\mapsto \varphi(c)\,\varphi(d) = vx\,. \end{aligned}$$

Thus, an edge is present in G_1 if and only if φ maps it to an edge in G_2, and hence $G_1 \simeq G_2$.

Example 8.4.3. We illustrate now the notion of graph isomorphism by listing all non-isomorphic simple graphs on n vertices, for $n = 1, 2, 3$. These are shown in Figure 8.10.

- For $n = 1$, there is only one simple graph: a single isolated vertex.
- For $n = 2$, there are two non-isomorphic simple graphs: one with no edges, and one with a single edge joining the two vertices.
- For $n = 3$, there are four non-isomorphic simple graphs. These correspond to having 0, 1, 2, or 3 edges. No two of these graphs are isomorphic, since they have different degree sequences.

Thus, up to isomorphism, there are 1, 2, and 4 simple graphs on 1, 2, and 3 vertices, respectively.

Figure 8.10: Non-isomorphic simple graphs on $n = 1, 2, 3$ vertices.

8.5 Complements

Given a simple graph, it is often useful to look not only at the edges that are present, but also at the edges that are *missing*. This idea leads to the notion of the **complement of a graph**:

Definition 8.5.1. *Let $G = (V, E)$ be a simple graph. The* ***complement*** *of G, denoted by $\bar{G}$, is the graph $\bar{G} = (V, F)$ with the same vertex set V, but with edge set F given by*

$$F = \Big\{ \{u, v\} \subset V : u \neq v \text{ and } \{u, v\} \notin E \Big\}.$$

In other words, $F = \binom{V}{2} \setminus E$ (recall (8.1)).

Informally, $\bar{G}$ has the edges that are missing from G.

Example 8.5.2. Let $G = (V, E)$ be the graph with vertex set $V = \{a, b, c, d\}$ and edge set $E = \{ac, ad, cd\}$. The complement $\bar{G}$ has the same vertex set, and contains precisely those edges between distinct vertices that are not present in G. In this case, $F = \{ab, bc, bd\}$. The graphs G and $\bar{G}$ are shown in the following Figure 8.11.

Figure 8.11: The graph G described on Example 8.5.2 (on the left) and its complement $\bar{G}$ (on the right).

8.6 Subgraphs

Often, when working with a graph, we are interested not in the entire graph, but only in a part of it: this leads to the notion of a **subgraph**.

Definition 8.6.1. *Let $G = (V, E)$ be a simple graph. A graph $G_1 = (V_1, E_1)$ is a* ***subgraph*** *of G if*

$$V_1 \subseteq V \quad \text{and} \quad E_1 \subseteq E.$$

In other words, a subgraph of G is obtained by deleting some vertices and/or edges from the original graph G.

Example 8.6.2. Let $G = (V, E)$ be the graph with vertex set $V = \{u, v, w, x\}$ and edge set $E = \{uv, ux, vx, vw\}$. Consider the graph $G_1 = (V_1, E_1)$ with $V_1 = \{u, v, x\}$ and $E_1 = \{uv, vx\}$. Since $V_1 \subseteq V$ and $E_1 \subseteq E$, the graph G_1 is a subgraph of G. The graphs G and G_1 are shown in Figure 8.12.

Figure 8.12: The graph G (on the left) and its subgraph G_1 (on the right), as defined in Example 8.6.2.

Chapter 9. Eulerian graphs

In Chapter 8, we restricted our attention to simple graphs, in which there is at most one edge between any pair of vertices.

! In this chapter, we temporarily relax this restriction and allow **multiple edges between the same pair of vertices**; graphs of this type are called multigraphs.

Definition 9.0.1. *A **multigraph** is a graph in which more than one edge is allowed between the same pair of vertices. The **degree** of a vertex is defined as before; as the number of edges incident with the vertex.*

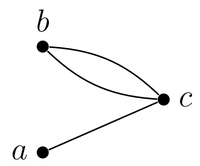

Figure 9.1: A multigraph $G = (V, E)$ with vertex set $V = \{a, b, c\}$ where the vertices b and c are joined by two distinct edges.

Example 9.0.2. In Figure 9.1 we see a multigraph $G = (V, E)$ with vertex set $V = \{a, b, c\}$. We may denote these edges by $e_1 = ac$, $e_2 = bc$, and $e_3 = bc$, so that the edge set is $E = \{e_1, e_2, e_3\}$. Although the edges e_2 and e_3 have the same endpoints, they are regarded as distinct edges.

In this graph, the degrees of the vertices are $d(a) = 1$, $d(b) = 2$, and $d(c) = 3$, since each edge is counted separately when computing degrees.

$\longrightarrow$ All graphs considered in this chapter are still **undirected** and have **no loops**.

9.1 Travelling in graphs

In graph theory, we are often interested in whether it is possible to *move through* a graph in a certain way. For instance, we may ask if we can travel from one vertex to another, traverse every edge exactly once, or return to our starting point without retracing our steps. To formalise these questions, we introduce several ways of travelling through a graph. These notions differ in how restrictive they are and form a natural hierarchy.

Definition 9.1.1. *Let $G = (V, E)$ be a graph. A **walk** γ in G is a finite sequence of vertices and edges*

$$\gamma = (w_0, e_1, w_1, e_2, \dots, e_k, w_k)\,,$$

where $w_0, .., w_k \in V$ and $e_1, .., e_k \in E$, such that for $1 \leq i \leq k$, the edge e_i joins the vertices w_{i-1} and w_i.

*The **length** of the walk is the number of edges it contains.*

*If the starting vertex w_0 is the same as the ending vertex w_k, the walk is said to be **closed**.*

Example 9.1.2. In the multipgraph from Figure 9.1, the walk $\gamma = (b, e_2, c, e_3, b)$ has length 2 and is closed.

Remark 9.1.3. In a simple graph, a walk is completely determined by its sequence of vertices, since there is at most one edge between any pair of vertices. In this case, we may describe a walk simply by writing

$$\gamma = (w_0, w_1, \dots, w_k).$$

Note that we do not require the vertices or edges of a walk to be distinct; that is, a walk may revisit vertices or travel along the same edge more than once. We now introduce more restrictive types of walks, in which repetitions of edges or vertices are not allowed:

Definition 9.1.4. *Let $G = (V, E)$ be a graph, and let $\gamma = (w_0, e_1, w_1, e_2, \ldots, e_k, w_k)$ be a walk in G. We say that γ is:*

- *a **trail** if no edge is repeated;*
- *a **circuit** if it is a closed trail of positive length, i.e. if $w_0 = w_k$ and $k \geq 1$;*
- *a **path** if it is a trail in which no vertex is repeated;*
- *a **cycle** if it is a closed path, that is, a circuit with no repeated vertices.*

Remark 9.1.5. Notice that vertices may be repeated in a trail, but not in a path. In contrast, edges may not be repeated in either a trail or a path.

Definition 9.1.6. *Let $G = (V, E)$ be a graph and $u, v \in V$. A walk $\gamma = (w_0, e_1, w_1, \ldots, e_k, w_k)$ in G is said to be* from u to v *if $w_0 = u$ and $w_k = v$.*

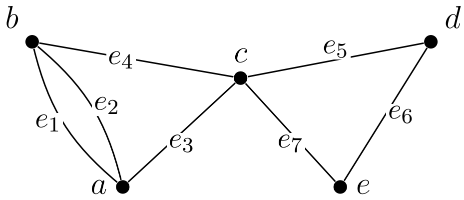

Figure 9.2: A multigraph $G = (V, E)$ with vertex set $V = \{a, b, c, d, e\}$.

Example 9.1.7. Consider the multigraph G in Figure 9.2, with edges labelled $e_1, \ldots, e_7$ as in the figure.

- The walk $\gamma_1 = (b, e_4, c, e_5, d)$ is a walk from b to d of length 2.
- The walk $\gamma_2 = (a, e_3, c, e_5, d, e_6, e)$ is a path from a to e.
- The walk $\gamma_3 = (a, e_1, b, e_4, c, e_7, e, e_6, d, e_5, c)$ is a trail, but not a path, since the vertex c is repeated.
- The walk $\gamma_4 = (a, e_1, b, e_4, c, e_5, d, e_6, e, e_7, c, e_3, a)$ is a circuit, but not a cycle, since the vertex c is visited more than once.
- The walk $\gamma_5 = (c, e_5, d, e_6, e, e_7, c)$ is a cycle.

Definition 9.1.8. *Let $G = (V, E)$ be a graph. A **Hamiltonian cycle** in G is a cycle that visits every vertex of G exactly once. A graph G is said to be **Hamiltonian** if it contains a Hamiltonian cycle.*

Lemma 9.1.9. *Let $G = (V, E)$ be a graph and let $u \neq v$ be vertices in V. If there is a walk in G from u to v, then there is a path in G from u to v.*

Proof. Let $\gamma = (w_0, e_1, w_1, \ldots, e_k, w_k)$ be a walk from u to v, so $w_0 = u$ and $w_k = v$.

If the vertices $w_0, w_1, \ldots, w_k$ are all distinct, then γ is already a path.

Otherwise, some vertex is repeated. Let w_i be the first vertex in the sequence $w_0, w_1, \ldots, w_k$ that appears more than once, and let w_j be its last occurrence on the walk, so that $i < j$ and $w_i = w_j$. Consider the shorter walk γ' obtained by deleting the portion of the walk between these two occurrences, namely

$$\gamma' = (w_0, e_1, w_1, \ldots, w_{\boldsymbol{i}}, e_{\boldsymbol{j+1}}, w_{\boldsymbol{j+1}}, \ldots, e_k, w_k).$$

This is still a walk from u to v, and it uses fewer vertices (and hence has fewer repetitions) than γ. Repeating this deletion process whenever a vertex is revisited, we eventually obtain a walk from u to v with no repeated vertices, that is, a path from u to v. ☐

Remark 9.1.10. The same argument shows that if G contains a circuit, then it contains a cycle.

Lemma 9.1.11. *If $G = (V, E)$ is a finite graph in which every vertex has degree at least 2, then G contains a cycle.*

Proof. Suppose $|V| = n$ and choose any vertex $u \in V$. We construct a walk in G strating from u. Since $d_G(u) \geq 2$, there exists a vertex $v \in V$ such that $\{u, v\} \in E$. Set $w_0 = u$, $w_1 = v$, and $e_1 = uv$; this will be the beginning of our construction.

Now suppose that we have constructed the following trail γ_m (*i.e.*, with no repeated edges):

$$\gamma_m = (w_0, e_1, w_1, \dots, e_m, w_m) .$$

If $w_m = w_0$, then γ_m is a circuit, and hence G contains a cycle.

If $w_m \neq w_0$, then since $d(w_m) \geq 2$, there exists an edge $e_{m+1} = \{w_m, w_{m+1}\} \in E$ with $e_{m+1} \neq e_m$. We may therefore extend the trail to $\gamma_{m+1} = (w_0, e_1, w_1, \dots, w_m, e_{m+1}, w_{m+1})$.

There are now two possibilities:

- The vertex w_{m+1} was already visited, that is, there exists $0 \leq i < m$ such that $w_{m+1} = w_i$. In this case, the subwalk $(w_i, e_{i+1}, w_{i+1}, \dots, e_{m+1}, w_{m+1})$ is closed and has no repeated edges, hence it is a circuit. Therefore, G contains a cycle.

- The vertex w_{m+1} has not yet been visited. In this case, γ_{m+1} is a trail of length $m + 1$.

Since G has only n vertices, we cannot encounter more than n distinct vertices without repetition. Therefore, after at most n steps, the first case must occur, and G contains a cycle. $\square$

9.2 Connectedness

We are often interested in knowing whether a graph forms a single whole, or whether it naturally breaks apart into separate parts. For instance, we may ask whether it is possible to travel between any two vertices of a graph, or whether some vertices are completely isolated from others. This leads to the notion of connectedness (or connectivity):

Definition 9.2.1. *A graph $G = (V, E)$ is said to be **connected** if, for any two vertices $u, v \in V$, there exists a walk in G from u to v.*

Even if a graph is not connected, parts of it may be, this leads into the following notion:

Definition 9.2.2. *Let $G = (V, E)$ be a graph. A **connected component** of G is a subgraph $G_1 = (V_1, E_1)$ such that:*

- *G_1 is connected;*
- *whenever $u, v \in V_1$ and $\{u, v\} \in E$, then $\{u, v\} \in E_1$;*
- *for any $u \in V_1$ and $v \in V \setminus V_1$, there is no walk in G from u to v.*

In other words, a connected component of G is a connected subgraph that is not contained in any larger connected subgraph of G.

Example 9.2.3. The graph in Figure 8.1 is not connected, since there is no path connecting the vertex e to any other vertex of the graph; one of its connected components is the subgraph containing the vertices $\{a, b, c, d\}$. In contrast, the graph in Figure 9.2 is connected.

The following properties follow directly from the previous Definition 9.2.2:

- Two vertices of G lie in the same connected component if and only if there exists a path in G between them.

- If a graph G is not connected, then it has at least two connected components.

9.3 The bridges of Königsberg

One of the questions a tourist might ask when visiting a new city is whether it is possible to visit all the landmarks without passing along the same streets more than once. This is precisely the kind of question that the residents of Königsberg asked themselves during their Sunday walks through the city. In the 18th century, this seemingly simple problem led to one of the earliest applications of graph theory.

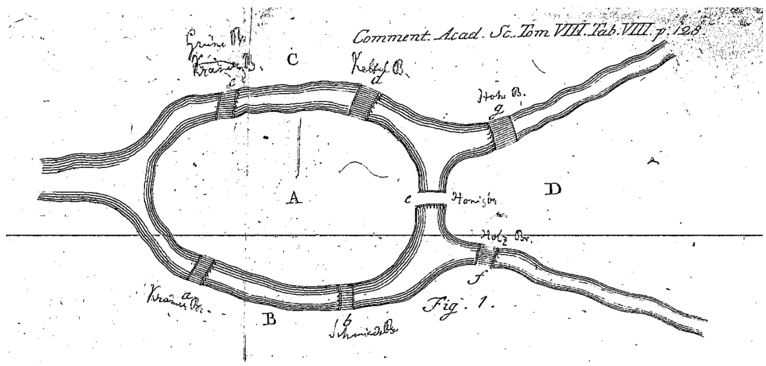

Figure 9.3: The bridges of Königsberg in Euler's 1741 work *Solutio problematis ad geometriam situs pertinentis*.

Königsberg was a city in East Prussia, divided into four regions by the Pregel river and connected by seven bridges, as shown in Figure 9.3. Their inhabitants would often ask themselves the following question:

> "Can one arrange a course in such a way as to pass over each bridge once and not more than once?"

Now, if we represent each land mass by a vertex and each bridge by an edge, we obtain a graph as shown in Figure 9.4, and this becomes a graph theory question: for a given graph, does there exist a walk that uses each edge exactly once?

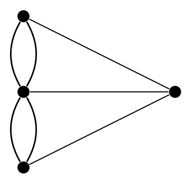

Figure 9.4: Graph representation of the Bridges of Königsberg.

Euler showed that such a walk in Königsberg is impossible. The key observation is that, in any closed walk that uses each edge exactly once, every time one enters a vertex one must also leave it. Consequently, each vertex must have even degree. In the graph corresponding to the Bridges of Königsberg, all four vertices have odd degree, and so no such walk can exist. This problem motivates the following definitions:

Definition 9.3.1. *An* ***Eulerian circuit*** *in a graph G is a circuit that uses each edge of G exactly once. A graph that contains an Eulerian circuit is called an* ***Eulerian graph****.*

9.4 Euler's theorem

The problem of the bridges of Königsberg motivated Euler to study, in general, when it is possible to traverse every edge of a graph exactly once and return to the starting point. His answer is given by the following theorem, which carries his name and characterises precisely when such a walk exists.

Theorem 9.4.1. *Let $G = (V, E)$ be a finite connected graph with $|V| > 1$. Then G has an Eulerian circuit if and only if every vertex of G has even degree.*

We have already proved the forward implication ($\Rightarrow$) of Theorem 9.4.1: in a closed walk that uses each edge exactly once, every time we enter a vertex we must also leave it, and hence every vertex must have even degree. To prove the converse ($\Leftarrow$), we will make use of the notion of *edge-disjoint* cycles and an auxiliary lemma.

Definition 9.4.2. *Let $G = (V, E)$ be a graph. Two cycles C_1 and C_2 in G are said to be* ***edge-disjoint*** *if they do not share any edges. More generally, a collection of cycles $C_1, C_2, \ldots, C_n$ in G is called* ***edge-disjoint*** *if no edge of G belongs to more than one of these cycles.*

Lemma 9.4.3. *Let $G = (V, E)$ be a finite connected graph with $|V| > 1$. If every vertex of G has even degree, then G can be written as a finite union of edge-disjoint cycles.*

Proof. Since G is connected and $|V| > 1$, it has no isolated vertices. Thus every vertex has degree at least 2, and by Lemma 9.1.11 the graph G contains a cycle C_1.

Remove all edges of C_1 from G. Each vertex of C_1 loses exactly two incident edges, while all other vertices are unchanged; in particular, all remaining vertices still have even degree. Discard any vertices that have become isolated.

If edges remain, the resulting graph is again finite and all its vertices have even degree, so by Lemma 9.1.11 it contains another cycle C_2. Repeating this process, and using the fact that G is finite, we eventually remove all edges of G.

Thus every edge of G belongs to exactly one of the cycles $C_1, C_2, \ldots, C_k$, and G is a union of edge-disjoint cycles. □

Proof of Theorem 9.4.1 ($\Longleftarrow$). Let G be a finite connected graph in which every vertex has even degree. By Lemma 9.4.3, the graph G is a union of k edge-disjoint cycles. We prove by induction on k that these cycles can be combined into a single Eulerian circuit.

Base case. If $k = 1$, then G itself is a single cycle, which is an Eulerian circuit. Indeed, a cycle by itself is already an Eulerian graph, since it provides a closed walk that uses each of its edges exactly once.

Induction hypothesis. Assume that any finite connected graph whose edges can be written as a union of at most $k - 1$ edge-disjoint cycles has an Eulerian circuit.

Induction step. Suppose that

$$G = C_1 \cup C_2 \cup \cdots \cup C_{k-1} \cup C_k$$

is a union of k edge-disjoint cycles, and let

$$H = C_1 \cup C_2 \cup \cdots \cup C_{k-1}.$$

The graph H may not be connected, so let $H_1, \ldots, H_j$ be its connected components. Each cycle C_i (for $1 \leq i \leq k-1$) lies entirely within a single connected component of H. Consequently, each H_i is a union of fewer than k edge-disjoint cycles and has all vertices of even degree. By the induction hypothesis, each H_i admits an Eulerian circuit.

Since G is connected, the cycle C_k must intersect every connected component H_i of H, each of which admits an Eulerian circuit. We now construct an Eulerian circuit for G as follows. Begin by traversing the cycle C_k. Whenever we first reach a vertex belonging to a component H_i, we temporarily leave C_k, traverse the Eulerian circuit of H_i in full, and then return to C_k at the same vertex. Continuing in this way, we obtain a single closed walk that uses every edge of G exactly once, and hence an Eulerian circuit. □

Chapter 10. Planar graphs

10.1 The three utilities problem

Let us begin with a classical puzzle, known as the *three utilities problem*, which will motivate the material studied in this chapter. Given three houses, the puzzle consists of determining whether it is possible to connect each house to the gas, water, and electricity supplies in such a way that none of the pipes or wires cross. Figure 10.1 illustrates this situation.

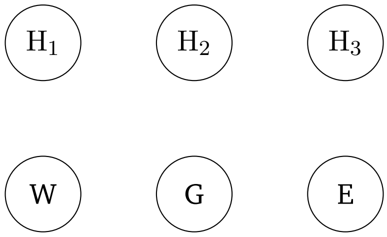

Figure 10.1: The three utilities problem.

At first sight, this problem looks like a puzzle about clever drawing. However, it can be translated into a precise question in graph theory.

Notice that we can distinguish two different kinds of vertices in this graph: the vertices corresponding to houses and the vertices corresponding to utility supplies. Moreover, each vertex of one kind needs to be connected to all vertices of the other kind. This motivates the following definitions.

Definition 10.1.1. *A simple graph $G = (V, E)$ is called* ***bipartite*** *if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge of G has one endpoint in V_1 and the other in V_2. In other words,this means that there exist two vertex subsets $V_1, V_2 \subseteq V$ such that*

$$V_1 \cup V_2 = V \qquad \text{and} \qquad V_1 \cap V_2 = \emptyset,$$

and every edge $e \in E$ connects a vertex in V_1 with a vertex in V_2.

If we want *every* vertex of V_1 to be connected to *every* vertex of V_2, then we obtain what is called a complete bipartite graph:

Definition 10.1.2. *Let $G = (V, E)$ be a bipartite graph with vertex partition $V = V_1 \cup V_2$. We say that G is a* ***complete bipartite graph*** *if every vertex in V_1 is adjacent to every vertex in V_2. In particular, if $|V_1| = m$ and $|V_2| = n$, the resulting graph is denoted by $K_{m,n}$ and has $m \times n$ edges.*

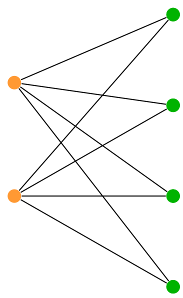

Figure 10.2: The complete bipartite graph $K_{2,4}$, with the two parts of the vertex set shown in different colours.

The graph associated to the three utilities problem is therefore the complete bipartite graph $K_{3,3}$.

We are now ready to introduce the notion of a planar graph.

Definition 10.1.3. *A graph is called* ***planar*** *if it can be drawn in the plane* ($\mathbb{R}^2$)*in such a way that no two edges cross, except possibly at their endpoints. Such a drawing is called a* ***planar representation*** *of the graph.*

Example 10.1.4. The complete graph K_4 is planar. One possible planar representation of K_4 is shown in Figure 10.3.

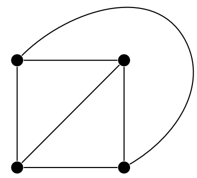

Figure 10.3: A planar representation of K_4.

With these definitions, the three utilities problem can be reformulated as the following graph-theoretic question: *is the graph $K_{3,3}$ planar?* In the remainder of this chapter, we shall develop tools that allow us to answer this question.

10.2 Faces and boundaries

To study planar graphs more systematically, we need to understand how a planar drawing of a graph divides the plane into regions, that we will call faces.

Definition 10.2.1. *Let $G = (V, E)$ be a simple planar graph and fix a planar drawing of G.*

- *The* ***faces*** *of the drawing are the connected regions of the plane that remain after the vertices and edges of the graph have been drawn. We denote by F the set of faces of the drawing, and by $f \in F$ a face.*
- *The* ***boundary*** *of a face f is the set of edges of G that lie along the border of f. We denote the boundary of $f \in F$ by $B(f)$.*
- *The* ***length*** *of the boundary of f, denoted by $b(f)$, is the number of edges on its boundary, counted with multiplicity if an edge appears more than once.*

In a planar drawing, every edge has two sides. If both sides of an edge belong to the same face, then the edge appears twice along the boundary of that face. This situation occurs precisely when the edge is a bridge.

Definition 10.2.2. *An edge e of a graph G is called a* ***bridge*** *if removing e increases the number of connected components of G.*

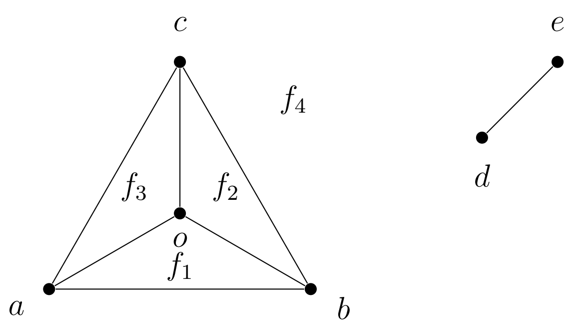

Figure 10.4: Planar representation of a disconnected graph $G = (V, E)$ with vertex set $V = \{a, b, c, d, e, o\}$.

Example 10.2.3. The planar graph G represented in Figure 10.4 has face set $F = \{f_1, f_2, f_3, f_4\}$ with boundaries

$$B(f_1) = \{ab, ao, bo\}, \quad B(f_2) = \{bc, bo, co\}, \quad B(f_3) = \{ac, ao, co\}, \quad B(f_4) = \{ab, bc, ac, de\}.$$

Notice that the edge de is a bridge; indeed, if we remove it, then the graph would have one connected component instead of two. Equivalently, both sides of the edge lie in the same face, namely f_4. Thus, the lengths of the boundaries are given by

$$b(f_1) = b(f_2) = b(f_3) = 3, \qquad b(f_4) = 5.$$

Remark 10.2.4. Every planar drawing has exactly one unbounded (infinite) face, corresponding to the region of the plane outside the graph; this follows from what is known as the Jordan Curve Theorem. In the drawing of Figure 10.4, the face f_4 is the unbounded face.

Just as the degrees of vertices in a graph are related to the number of edges via the handshaking lemma, there is an analogous relationship in planar graphs between the edges and the boundaries of faces. This is known as the planar handshaking lemma:

Lemma 10.2.5 (Planar handshaking lemma). *Let $G = (V, E)$ be a planar graph with face set F. Then*

$$\sum_{f \in F} b(f) = 2|E|. \tag{10.1}$$

Proof. Each edge in a planar drawing has two sides, and each side lies on the boundary of a face. Thus each edge contributes 1 to the boundary length of two (not necessarily distinct) faces, and so contributes 2 in total. Summing over all edges gives the desired equality (10.1). □

The following two results give tools to recognise bridges in graphs.

Lemma 10.2.6. *Let $G = (V, E)$ be a simple graph. An edge $e \in E$ is a bridge if and only if it is not contained in any cycle of G.*

Proof. Let e join the vertices u and v in the vertex set V.

($\Rightarrow$) Suppose that e is not a bridge. Then removing e does not disconnect the graph (by definition of a bridge), so there is still a path from u to v that does not use e. Together with the edge e, this path forms a cycle containing e.

($\Leftarrow$) Conversely, suppose that e is contained in a cycle. Then there is a path from u to v that does not use e. Removing e therefore does not disconnect the graph, and so e is not a bridge. □

Notice that Lemma 10.2.6 holds for all graphs and does not require planarity. Planarity is only used in the following Corollary 10.2.7, which aims to recognise bridges in planar graphs.

Corollary 10.2.7. *In a planar representation of a finite graph, an edge has both sides in the same face if and only if it is a bridge.*

Proof. ($\Rightarrow$) Suppose that an edge e is not a bridge. By Lemma 10.2.6, e is contained in a cycle. In a planar drawing, this cycle forms a closed curve, and the two sides of the edge lie on opposite sides of this curve. Hence the two sides of e lie in different faces, so e cannot have both sides in the same face.

($\Leftarrow$) Conversely, suppose that e is a bridge joining vertices u and v. Removing e disconnects the graph into two components, one containing u and the other containing v. In the planar drawing, it is therefore possible to move from one side of e to the other by travelling around the outside of one of these components without crossing any edge. Hence both sides of e lie in the same face. □

10.3 Euler's Formula

$\longrightarrow$ From now on, for simplicity, when we say "planar graph" we mean a fixed planar representation of a graph.

This section is dedicated to Euler's formula, a fundamental relation between the numbers of vertices, edges and faces of a planar graph, and one of the key tools in planar graph theory.

Theorem 10.3.1 (Euler's formula). *Let $G = (V, E)$ be a finite planar graph with face set F and k connected components. Then*

$$|V| - |E| + |F| = k + 1.$$

Proof. We prove the result by induction on the number of edges $n = |E|$.

Base case. If $|E| = n = 0$, then G consists of $k = |V|$ isolated vertices (since G has k connected components), and there is exactly one face, namely the unbounded face. Thus

$$|V| - |E| + |F| = |V| - 0 + 1 = k + 1,$$

so the formula holds in this case.

Induction hypothesis. Assume that Euler's formula holds for all planar graphs with $n - 1$ edges.

Induction step. Let G be a planar graph with n edges, and remove one edge e to obtain a planar graph G'. Then G' has the same vertices, $n - 1$ edges, and some number of faces $|F'|$ and components k'.

By the induction hypothesis,

$$|V| - (n - 1) + |F'| = k' + 1.$$

We now compare G and G' by considering whether e is a bridge.

- If e is not a bridge, then removing it does not change the number of connected components, so $k' = k$. By Corollary 10.2.7, the two sides of e lie in different faces of G, which merge into a single face in G'. Hence $|F'| = |F| - 1$. Substituting into the induction equation gives

  $$|V| - (n - 1) + (|F| - 1) = k + 1,$$

  which simplifies to $|V| - n + |F| = k + 1$.

- If e is a bridge, then removing it increases the number of connected components by one, so $k' = k+1$. Again by Corollary 10.2.7, both sides of e lie in the same face, so removing e does not change the number of faces: $|F'| = |F|$. Substituting into the induction equation gives

  $$|V| - (n - 1) + |F| = (k + 1) + 1,$$

  which again simplifies to $|V| - n + |F| = k + 1$.

In both cases, this completes the induction and proves Euler's formula. $\square$

Remark 10.3.2. In particular, if G is a connected graph, then Euler's formula reduces to

$$|V| - |E| + |F| = 2.$$

Example 10.3.3. Consider the planar graph in Figure 10.4. It has vertex set $V = \{a, b, c, d, e, o\}$, so $|V| = 6$. The edge set is given by $E = \{ab, bc, ca, ao, bo, co, de\}$, hence $|E| = 7$. The planar representation has four faces, labelled $F = \{f_1, f_2, f_3, f_4\}$, so $|F| = 4$. Finally, the graph has two connected components (the triangle, and the single edge de), so $k = 2$. Thus Euler's formula gives

$$|V| - |E| + |F| = 6 - 7 + 4 = 3 = k + 1.$$

10.4 Simple planar graphs have few edges

If we allow multiple edges between the same pair of vertices, then there is no upper bound on the number of edges of a planar graph (see Figure 10.5).

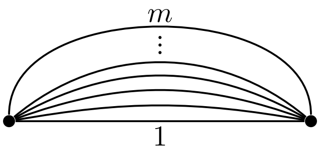

Figure 10.5: A planar graph with m edges.

However, for simple graphs the situation is very different. In general, a simple graph with $|V|$ vertices can have at most $\frac{|V|(|V|-1)}{2}$ edges. This naturally leads to the following question: what is the maximum number of edges that a simple planar graph can have?

As we shall see, the answer is surprisingly small. In particular, in this section we will show that if a simple graph has too many edges, then it cannot be planar.

We begin with a basic upper bound on the number of edges of a planar graph.

Theorem 10.4.1. *If $G = (V, E)$ is a simple planar graph with $|V| \geq 3$, then*

$$|E| \leq 3|V| - 6. \tag{10.2}$$

Proof. If $|E| \leq 2$, then the inequality holds trivially since $|V| \geq 3$.

Assume now that $|E| \geq 3$. Since G is simple, it has no loops or multiple edges, so no face boundary can have length 1 or 2. Hence for every face $f \in F$ we have $b(f) \geq 3$. Therefore, by the planar handshaking lemma,

$$2|E| = \sum_{f \in F} b(f) \geq 3|F|,$$

and thus $|F| \leq \frac{2}{3}|E|$.

Now using Euler's formula,

$$|V| - |E| + |F| = k + 1 \geq 2,$$

and substituting the bound on $|F|$ gives

$$|V| - |E| + \frac{2}{3}|E| \geq 2.$$

Rearranging the terms, one obtains (10.2). □

This is useful because it allows us to determine whether a graph is planar without drawing it. For instance:

Corollary 10.4.2. *The complete graph K_5 is not planar.*

Proof. The graph K_5 has $|V| = 5$ and $|E| = 10$. But

$$3|V| - 6 = 9 < 10 = |E|,$$

so K_5 does not satisfy the inequality in Theorem 10.4.1. Hence it is not planar. □

Returning to the three utilities problem, the argument in Theorem 10.4.1 does not rule out the possibility that $K_{3,3}$ is planar, since $|E| = 9 \leq 12 = 3|V| - 6$. We now refine the argument to deal with this case.

Definition 10.4.3. *The* ***girth*** *g of a graph G is the length of its shortest cycle. If G has no cycles, then $g = 0$.*

Proposition 10.4.4. *Let $G = (V, E)$ be a simple planar graph with girth $g \geq 3$. Then*

$$|E| \leq \frac{g}{g-2}(|V| - 2).$$

Proof. By definition of faces and girth, for every face $f \in F$ we have $b(f) \geq g$, that is, every face is bounded by at least g edges. Together with the planar handshaking lemma, this gives

$$2|E| = \sum_{f \in F} b(f) \geq g|F|,$$

and hence $|F| \leq \frac{2|E|}{g}$.
Using Euler's formula, and since there is at least one connected component, we have

$$|V| - |E| + |F| = k + 1 \geq 2.$$

Substituting the bound on $|F|$ yields

$$|V| - |E| + \frac{2|E|}{g} \geq 2.$$

Finally, rearranging gives

$$|E| \leq \frac{g}{g-2}(|V| - 2).$$

□

Remark 10.4.5. Since G is simple, it has no loops or multiple edges, and therefore its girth cannot be 1 or 2. In particular, if $g = 3$, the bound above reduces to $|E| \leq 3|V| - 6$, recovering Theorem 10.4.1.

We are now ready to solve the three utilities problem:

Corollary 10.4.6. *The graph $K_{3,3}$ is not planar.*

Proof. First note that a bipartite graph has no cycles of odd length. Indeed, since edges only join vertices in different parts of the bipartition, vertices along any path alternate between the two parts, and a path can return to its starting vertex only after an even number of steps.

Since the graph $K_{3,3}$ is bipartite, it has no odd cycles, and therefore its girth is $g = 4$. As $|V| = 6$, Proposition 10.4.4 would give

$$|E| \leq \frac{4}{2}(6 - 2) = 8,$$

which is false since $|E| = 9$. Hence $K_{3,3}$ is not planar. □

We now give a further simple but important consequence of the edge bound for planar graphs.

Theorem 10.4.7. *Every finite simple planar graph has a vertex of degree at most 5.*

Proof. Assume that every vertex has degree at least 6. Then by the handshaking lemma,

$$2|E| = \sum_{v \in V} d(v) \geq 6|V|,$$

so $|E| \geq 3|V|$, which contradicts the inequality in Theorem 10.4.1. Hence the graph cannot be planar. □

Finally, we introduce a notion that allows us to describe planarity in a more general way, and state a classical theorem that characterises planar graphs.

Definition 10.4.8. *A **subdivision** of a graph G is obtained by replacing edges of G by paths, that is, by inserting new vertices of degree 2 along edges.*

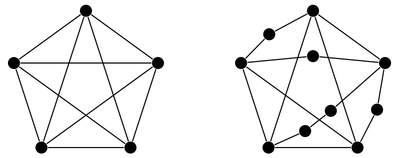

Figure 10.6: On the left, the graph K_5. On the right, a subdivision of K_5 obtained by inserting vertices of degree 2 along some edges.

Remark 10.4.9. Subdividing an edge does not affect whether a graph can be drawn in the plane without crossings.

Theorem 10.4.10 (Kuratowski's Theorem). *A finite graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ as a subgraph.*

We will not prove this theorem in this course. The forward implication follows from the fact that planarity is preserved under subdivisions, together with the non-planarity of K_5 and $K_{3,3}$. The converse direction is substantially more difficult.

10.5 Platonic graphs

We conclude this chapter with a classical and elegant application of Euler's formula. We ask the following question: how symmetric can a planar graph be?

Definition 10.5.1. *A **Platonic graph** is a finite, simple, connected planar graph in which*

- *every vertex has the same degree $d \geq 3$, and*
- *every face has the same boundary length $b \geq 3$.*

These graphs correspond exactly to the graphs of the Platonic solids.

Theorem 10.5.2. *There are exactly five Platonic graphs.*

Proof. Let $G = (V, E)$ be a Platonic graph, where every vertex has degree d and every face has boundary length b.

By the handshaking lemma applied to the vertices, and the planar handshaking lemma applied to the faces, we have

$$\sum_{v \in V} d(v) = d|V| = 2|E| \quad \text{and} \quad \sum_{f \in F} b(f) = b|F| = 2|E|.$$

Hence $d|V| = 2|E| = b|F|$.

Since G is connected, Euler's formula gives $|V| - |E| + |F| = 2$. Substituting $|E| = \frac{d|V|}{2}$ and $|F| = \frac{d|V|}{b}$ yields

$$|V| - \frac{d|V|}{2} + \frac{d|V|}{b} = 2.$$

Multiplying by $2b$ gives $|V|(2b - bd + 2d) = 4b$.

Since $|V| > 0$ and $b > 0$, this implies $2b + 2d - bd > 0$, which is equivalent to

$$(b-2)(d-2) < 4.$$

As $b \geq 3$ and $d \geq 3$ are integers, there are only five possible pairs (b, d) satisfying this inequality. Each of these gives rise to a Platonic graph; we list them in the following table.

d	b	$\lvert V \rvert$	$\lvert E \rvert$	$\lvert F \rvert$	Name
3	3	4	6	4	Tetrahedron
3	4	8	12	6	Cube (Hexahedron)
4	3	6	12	8	Octahedron
3	5	20	30	12	Dodecahedron
5	3	12	30	20	Icosahedron

$\square$

This result shows the strength of Euler's formula: from a simple counting argument, we obtain a complete classification of the most symmetric planar graphs.